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Random Response of Symmetric Cross-Ply Composite Beams with Arbitrary Boundary Conditions

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A generalized modal approach is presented to solve the equations of motion of a laminated composite beam obtained with a third-order shear deformation theory. The biorthonormal eigenfunctions of the differential equations expressed in the state form are used to decouple the equations. To obtain these eigenfunctions for beams with any arbitrary beam boundary conditions, a method is presented. The solution obtained by this approach is used to calculate the beam response for spatially and temporally correlated random loads. Several sets of numerical results are presented to demonstrate the importance of shear deformations in the dynamic analysis of composite beams.

Introduction

THE dynamic behavior of isotropic beams with shear deformations and rotatory inertia was initially examined by Timoshenko[1] in 1921, and since then, such beams have been called Timoshenko beams. There has been continued research interest in the vibration analysis of these beams and the literature is now replete with the publications on this subject.

The effect of shear deformations is more pronounced in the beams made with high-strength composite materials because of their low transverse shear rigidity. The transverse shear modulus values of 1/30 of the extensional modulus have been reported for some graphite epoxy composites. Therefore, proper consideration of the shear flexibility in the dynamic analysis of beams made with such materials is quite necessary.

Some notable studies in the free-vibration analysis of composite beams are by Dudek,[2] Abarcar and Cunniff,[3] Krishna Murthy and Shimpi,[4] Miller and Adams,[5] Teoh and Huang,[6] and more recently, Chandrashekhara et al.[7] Dudek[2] used the results of Timoshenko beam theory to obtain the transverse shear modulus and study the effect of the ratio of transverse shear and extensional moduli on the beam frequencies. Abarcar and Cunniff[3] showed the existence of coupling between the torsional modes and bending modes in an orthotropic beam experimentally and proposed a discrete model to analyze the free-vibration characteristics of a cantilever beam. Neglecting the shear deformations and rotatory inertia, Miller and Adams[5] developed the equations of motion of a generally orthotropic beam with coupled bending and torsional vibrations and obtained the free-vibration characteristics of beams with several different boundary conditions. Later Teoh and Huang[6] also included the shear deformation and rotatory inertia according to the Timoshenko beam theory in the torsional and flexural vibration study of an orthotropic cantilever beam. Krishna Murthy[8] and Krishna Murthy and Shimpi[4] were, perhaps, the first authors to propose the use of a so-called third-order theory to include the effect of shear deformation in the dynamic analysis of isotropic beams and laminated beams, respectively. More recently, Chandrashekhara et al.[7] have developed the equations of motion of composite beams using a first-order shear deformation theory and have obtained the frequencies and mode shapes of composite beams with several different boundary conditions. Suresh et al.[9] have studied the effect of assumed warping behavior in the formulation on the free-vibration characteristics of torsionally and flexurally coupled composite beams.

In the present paper, we examine the forced vibration response of composite beams. We consider a well-known third-order shear deformation theory (proposed in somewhat different forms by Krishna Murthy,[8,10] Levinson,[11] Bhimaraddi and Steven,[12] and Reddy[13,14] for isotropic, sandwiched, and composite plates) to obtain the equations of motion, which include the shear deformations and rotatory inertia terms. To solve the coupled equations of motion for arbitrary loading and arbitrary boundary conditions, a general modal analysis approach, utilizing the state form of the equations of motion and their biorthonormal eigenfunctions, is presented. This solution is used to obtain the random response of beams with different boundary conditions and parameters, subjected to spatially and temporally correlated random loads.

Analytical Formulation

To develop the equations of motion for a cross-ply symmetric laminated beam, the displacement field proposed in a third-order theory is assumed as

$$\hat{u}_1 = -\hat{z}\frac{\partial \hat{w}}{\partial \hat{x}} + \hat{z}\left[1 - \frac{4}{3}\left(\frac{\hat{z}}{h}\right)^2\right]\phi$$
$$\hat{u}_2 = 0 \tag{1}$$
$$\hat{u}_3 = \hat{w}$$

where $\hat{u}_1$, $\hat{u}_2$, and $\hat{u}_3$ are the displacements along the x, y, and z coordinates of the beam, respectively (see Fig. 1); $\hat{w}$ the

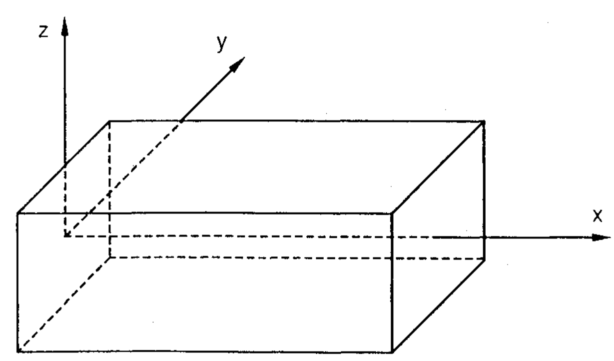

Fig. 1 Coordinate axes for the beam.

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displacement of a point on the midplane; $\psi = [\phi - (\partial \hat{w}/\partial \hat{x})]$ the rotation of the normal to the midplane; and h the beam thickness. This displacement field provides a parabolic variation of the transverse shear stress that is zero at the top and bottom surfaces of the beam.

The equations of motion and associated boundary conditions are obtained by using the variational approach.[15] The principle of virtual displacement for the beam can be written as

$$\delta\pi = \int_0^{\hat{t}}\left[\int_V(\hat{\sigma}_x\delta\epsilon_x + 2\hat{\sigma}_{xz}\delta\epsilon_{xz})\,\mathrm{d}V - \int_V\rho(\hat{\ddot{u}}_1\delta\hat{u}_1 + \hat{\ddot{u}}_3\delta\hat{u}_3)\,\mathrm{d}V - \int_0^L\hat{f}(\hat{x},\hat{t})\delta\hat{w}\,\mathrm{d}\hat{x}\right]\mathrm{d}\hat{t} \tag{2}$$

where $\hat{\sigma}_x$ and ϵ_x are the normal stress and strain, respectively; $\hat{\sigma}_{xz}$ and ϵ_{xz} the shear stress and strain, respectively; ρ the mass density; L the length of the beam; and $\hat{f}(\hat{x},\hat{t})$ the transverse load per unit length of the beam applied in the plane of symmetry. Substituting for the strains in terms of displacements and following the standard procedures of the variational techniques, we obtain the following equations of motion:

$$-\frac{\partial^2\hat{M}}{\partial\hat{x}^2} - \hat{I}_2\frac{\partial^4\hat{w}}{\partial\hat{t}^2\partial\hat{x}^2} + (\hat{I}_2 - h_2\hat{I}_4)\frac{\partial^3\phi}{\partial\hat{t}^2\partial\hat{x}} + \hat{I}_0\frac{\partial^2\hat{w}}{\partial\hat{t}^2} - \hat{f} = 0 \tag{3}$$

$$-\frac{\partial\hat{M}}{\partial\hat{x}} + h_2\frac{\partial\hat{P}}{\partial\hat{x}} + \hat{Q} - 3h_2\hat{S} + (\hat{I}_2 - h_2\hat{I}_4)\frac{\partial^3\hat{w}}{\partial\hat{t}^2\partial\hat{x}} + (\hat{I}_2 - 2h_2\hat{I}_4 + h_2^2\hat{I}_6)\frac{\partial^2\phi}{\partial\hat{t}^2} = 0 \tag{4}$$

and the following sets of the natural and essential boundary conditions, one of which must be specified:

$$\{\hat{M} - h_2\hat{P}:\phi\};\qquad \left\{-\hat{M}:\frac{\partial\hat{w}}{\partial\hat{x}}\right\}$$
$$\left\{\frac{\partial\hat{M}}{\partial\hat{x}} + \hat{I}_2\frac{\partial^3\hat{w}}{\partial\hat{t}^2\partial\hat{x}} - (\hat{I}_2 - h_2\hat{I}_4)\frac{\partial^2\phi}{\partial\hat{t}^2} : w\right\} \tag{5}$$

In Eqs. (3–5), the following force resultants, inertial, and other quantities have been introduced:

$$\hat{M} = \int_A\hat{z}\hat{\sigma}_x\,\mathrm{d}A,\qquad \hat{P} = \int_A\hat{z}^3\hat{\sigma}_x\,\mathrm{d}A$$
$$\hat{Q} = \int_A\hat{\sigma}_{xz}\,\mathrm{d}A,\qquad \hat{S} = \int_A\hat{z}^2\hat{\sigma}_{xz}\,\mathrm{d}A$$
$$\hat{I}_j = \int_A\rho\hat{z}^j\,\mathrm{d}A,\qquad j = 0, 1, \ldots, 6$$
$$h_2 = \frac{1}{3}\left(\frac{2}{h}\right)^2 \tag{6}$$

We will now write these equations in terms of the nondimensional displacement and other parameters. For this, we introduce the following quantities:

$$\hat{x} = Lx,\quad \hat{w} = Lw,\quad \hat{z} = hz,\quad \hat{t} = t\,(h/L^2)\sqrt{E_1/\rho} \tag{7}$$

where E_1 is a reference value of the Young's modulus that is taken to be the Young's modulus of a lamina in the strong direction. Expressing the strains in terms of the displacement derivatives, utilizing the stress-strain equations to define the stresses, substituting these into Eqs. (6) to define stress resultants in terms of displacement quantities, and then, finally, substituting for the stress resultants defined into Eqs. (3) and (4) and after some readjustments, we obtain the following nondimensional equations of motion:

$$w'''' = k_{56}\phi' + m_{51}\ddot{w} + m_{53}\ddot{w}'' + m_{56}\ddot{\phi}' + f(x,t) \tag{8}$$

$$\phi'' = k_{64}\phi + k_{65}w''' + m_{62}\ddot{w}' + m_{64}\ddot{\phi} \tag{9}$$

where a prime and dots on a quantity denote the spatial and time derivatives, respectively. The normalized force $f(x,t)$ and the coefficients k_{ij} and m_{ij} in Eqs. (8) and (9) are defined in the Appendix. The motivation behind writing the equations of motion in this particular form will be apparent soon.

For a forced vibration analysis of this problem defined by the coupled partial differential equations (8) and (9), we present a generalized modal approach. Later, we will use this approach to obtain the beam response for randomly defined lateral loads.

This problem, being a conservative one, can be solved by the normal mode approach. This approach will require the calculation of the natural frequencies, normal modes of vibration for each frequency, and the definition of the orthogonality equations for these normal modes. For this, first we solve a free-vibration problem. We assume the separation of variables, which requires that temporal variations be harmonic, say, with natural frequency ω. With temporal variations removed from Eqs. (8) and (9), each of these equations can be written as the sixth-order, homogeneous ordinary differential equations with ω as a parameter. By application of the boundary conditions (see Huang[16] for a Timoshenko beam, Miller and Adams[5] and Teoh and Huang[6] for torsionally and flexurally coupled orthotropic beams, and Chandrashekhara et al.[7] for composite beams) one can define the characteristic equation to obtain the frequencies and corresponding six coefficients of integration and, thus, the eigenfunctions. Most of the work reported in the literature on composite beams with general boundary conditions has stopped here. The next step, which is essential to solve a forced vibration problem by the normal modes approach, is the development of the orthogonality conditions for these normal modes. This development of the orthogonality conditions explicitly for a composite structure using a higher-order shear deformation theory can sometimes be involved. The approach that we present next avoids this explicit formulation of these conditions with minimum extra effort. This approach was initially used by the authors[17] for solving a Timoshenko beam problem.

In this approach, we write the equation of motion in the state form by introducing the following state variables:

$$w_1 = w \tag{10}$$

$$w_2 = \frac{\partial w}{\partial x} = \frac{\partial w_1}{\partial x} \tag{11}$$

$$w_3 = \frac{\partial^2 w}{\partial x^2} = \frac{\partial w_2}{\partial x} \tag{12}$$

$$w_4 = \phi \tag{13}$$

$$w_5 = \frac{\partial^3 w}{\partial x^3} = \frac{\partial w_3}{\partial x} \tag{14}$$

$$w_6 = \frac{\partial\phi}{\partial x} = \frac{\partial w_4}{\partial x} \tag{15}$$

Equations (11), (12), (14), and (15) along with Eqs. (8) and (9) can be combined into a single first-order system of equations in the spatial variable as

$$\{w'\} = [K]\{w\} + [M]\{\ddot{w}\} + \{F(x,t)\} \tag{16}$$

The k_{ij} and m_{ij} of matrices $[K]$ and $[M]$, some of which were introduced in Eqs. (8) and (9), are completely defined in the Appendix. The elements of the vectors $\{w\}$ and $\{F(x,t)\}$ are

$$\{w\}^T = (w_1, w_2, w_3, w_4, w_5, w_6)$$
$$\{F(x,t)\}^T = [0, 0, 0, 0, f(x,t), 0] \tag{17}$$

We will first solve Eq. (16) for free vibration to obtain the frequencies and eigenfunctions. For the solution to be separable in the time and spatial coordinates

$$\{w(x,t)\} = \{W(x)\}q(t) \tag{18}$$

$q(t)$ must satisfy a harmonic equation:

$$\ddot{q}(t) + \omega^2 q(t) = 0 \tag{19}$$

and $\{W\}$ must satisfy

$$\{W'\} = [K]\{W\} - \omega^2[M]\{W\} \tag{20}$$

The solution of the system of Eq. (20) can be written in the following form:

$$\{W\} = \sum_{j=1}^{6} c_j\{\phi_j\}e^{\lambda_j x} \tag{21}$$

where λ_j and $\{\phi_j\}$ are the eigenvalues and eigenvectors of the following eigenvalue problem associated with Eq. (20):

$$[K - \omega^2 M]\{\phi_j\} = \lambda_j\{\phi_j\} \tag{22}$$

Equation (22) is cubic in λ_j^2 as

$$\lambda_j^6 - a_1\lambda_j^4 + a_2^2\lambda_j^2 - a_3 = 0 \tag{23}$$

with

$$a_1 = k_{56}k_{65} + k_{64} - (m_{64} + k_{65}m_{56} + m_{53})\omega^2$$

$$a_2 = (k_{56}m_{62} - k_{64}m_{53} + m_{51})\omega^2 + (m_{53}m_{64} - m_{56}m_{62})\omega^4 \tag{24}$$

$$a_3 = -\omega^2 m_{51}(k_{64} - \omega^2 m_{64})$$

It is simple to obtain the roots of Eq. (23) for known values of the coefficients a_1, a_2, and a_3. It was observed that, for the values of the coefficients encountered in the numerical calculations, either there were four real and two imaginary roots or two real and four imaginary roots of Eq. (23). For each value of λ_j, the corresponding eigenvector $\{\phi_j\}$ can also be obtained in closed form as

$$\phi_{j1} = 1, \quad \phi_{j2} = \lambda_j, \quad \phi_{j3} = \lambda_j^2, \quad \phi_{j4} = \phi_{j6}/\lambda_j, \quad \phi_{j5} = \lambda_j^3$$
$$\phi_{j6} = [\lambda_j^4 + \omega^2(m_{51} - \lambda_j^2 m_{53})]/(k_{56} - \omega^2 m_{56}) \tag{25}$$

The elements of $\{W\}$ must, of course, satisfy the beam boundary conditions. The natural boundary conditions in terms of the stress resultants are given in Eqs. (5). They can be defined easily in terms of the displacement variables. For the standard cases of the fixed end, hinged end, and free end, they are as follows.

Fixed end:

$$\hat{w} = \phi = \frac{\partial \hat{w}}{\partial \hat{x}} = 0$$

which are equivalent to

$$\sum_{j=1}^{6} c_j\phi_{ji}e^{\lambda_j x} = 0, \qquad i = 1, 2, 4 \tag{26}$$

Hinged end:

$$\hat{w} = 0, \qquad \hat{M} - h_2\hat{P} = 0, \qquad h_2\hat{P} = 0$$

These are equivalent to

$$w = 0, \qquad \frac{\partial^2 w}{\partial x^2} = 0, \qquad \frac{\partial \phi}{\partial x} = 0$$

or

$$\sum_{j=1}^{6} c_j\phi_{ji}e^{\lambda_j x} = 0, \qquad i = 1, 3, 6 \tag{27}$$

Free end:

$$\hat{M} - h_2\hat{P} = 0, \qquad h_2\hat{P} = 0$$

$$\frac{\partial \hat{M}}{\partial \hat{x}} + \hat{I}_2\frac{\partial^3 \hat{w}}{\partial \hat{t}^2\partial \hat{x}} - (\hat{I}_2 - h_2\hat{I}_4)\frac{\partial^2 \phi}{\partial \hat{t}^2} = 0$$

which is equivalent to

$$\frac{\partial^2 w}{\partial x^2} = 0, \quad \frac{\partial \phi}{\partial x} = 0, \quad F_2\frac{\partial w}{\partial x} + F_4\phi - \frac{\partial^3 w}{\partial x^3} = 0$$

or

$$\sum_{j=1}^{6} c_j\phi_{ji}e^{\lambda_j x} = 0, \qquad i = 3, 6$$
$$\sum_{j=1}^{6} c_j(F_2\phi_{j2} + F_4\phi_{j4} - \phi_{j5})e^{\lambda_j x} = 0 \tag{28}$$

where

$$F_2 = -\omega^2 m_{53}, \qquad F_4 = k_{56} - \omega^2 m_{56} \tag{29}$$

In Eqs. (26–28), the variable x is either 0 or 1. It is also pointed out that the variable ϕ is related to the rotation of the normal, whereas ϕ_{jk} denotes the kth element of the eigenvector $\{\phi_j\}$. For partially restrained ends also, similar boundary conditions can be developed explicitly.

The six boundary conditions (three on each end) provide a set of six homogeneous simultaneous equations for calculating the coefficients c_j of Eq. (21) as

$$[\Delta]\{c\} = \{0\} \tag{30}$$

where c_j are the elements of $\{c\}$. The elements of $[\Delta]$ are defined by the coefficients of the two chosen sets of Eq. (26), (27), or (28), whichever ones are applicable to the beam boundary conditions.

For a nontrivial solution of Eq. (30), determinant of $[\Delta]$ must be zero. This provides the characteristic equation for the frequency parameter ω. This equation will have an infinite number of roots that can be obtained by a simple Newton-Raphson technique. Initial guesses can be obtained by a simple plotting of the function to locate the zero crossings.

Table 1 shows the frequencies obtained by this approach for a cross-ply (0, 90, 90, 0) AS/3501 graphite-epoxy laminated beam with the L/h ratio = 10 and four different combinations of boundary conditions.

For each frequency ω_n, there is a set of coefficients c_{nj} [obtained from Eq. (30)], eigenvalues λ_{nj} [obtained by solving Eq. (23)], and eigenvectors $\{\phi_{nj}\}$ [as defined by Eqs. (25)].

Table 1 Natural frequencies of (0, 90, 90, 0) beams with different boundary conditions

Mode number	Fixed-fixed	Fixed-hinged	Fixed-free	Hinged-hinged
1	3.7751	3.0447	0.8891	2.3189
2	8.0440	7.5593	4.1792	7.0171
3	12.998	12.565	9.1916	12.132
4	18.165	17.732	14.384	17.301
5	23.502	23.011	19.715	22.533
6	28.991	28.430	25.093	27.881
7	34.675	34.027	30.620	33.396
8	40.576	39.838	36.274	39.119
9	46.735	45.896	42.156	45.082
10	53.156	51.592	47.876	49.501

These are then used to define the eigenfunction $\{W_n\}$ according to Eq. (21) as

$$\{W_n\} = \sum_{j=1}^{6} c_{nj}\{\phi_{nj}\}e^{\lambda_{nj}x} \tag{31}$$

These eigenfunctions form a complete set. They can be used to express the response function vector $\{w\}$ as

$$\{w\} = \sum_{n=1}^{\infty} q_n(t)\{W_n\} \tag{32}$$

However, as these eigenfunctions do not form an orthogonal set, they cannot be used to decouple Eq. (16) to obtain $q_n(t)$. This would have been possible if Eq. (16) were a self-adjoint equation. To be able to decouple Eq. (16), we must therefore obtain the eigenfunction of the adjoint of Eq. (20).[18]

Adjoint Problem

It is simple to show (see Ref. 19) that the adjoint of Eq. (20) is

$$\{V'\} = -[K - \omega^2 M]^T\{V\} \tag{33}$$

with the boundary conditions defined according to the following equation:

$$\{V\}^T\{W\}\,|_0^1 = 0 \tag{34}$$

where $\{V\}$ is the adjoint vector. This vector can be expressed in terms of the eigenvalues μ_j and eigenvectors $\{\psi_j\}$ of Eq. (33) as

$$\{V\} = \sum_{j=1}^{6} d_j\{\psi_j\}e^{\mu_j x} \tag{35}$$

where μ_j and $\{\psi_j\}$ are obtained from

$$[K - \omega^2 M]^T\{\psi_j\} = -\mu_j\{\psi_j\} \tag{36}$$

It is simple to see from the comparison of Eqs. (22) and (36) that

$$\mu_j = -\lambda_j \tag{37}$$

From the solution of Eq. (36), the eigenvector elements are

$$\begin{aligned} &\psi_{j1} = -\omega^2 m_{51}\psi_{15}/\lambda_j, \quad \psi_{j2} = (\psi_{j1} - \omega^2 m_{62}\psi_{j6})/\lambda_j \\ &\psi_{j3} = \lambda_j\psi_{45} - k_{65}\psi_{j6}, \qquad \psi_{j4} = 1 \\ &\psi_{j5} = (\lambda_j\psi_{j6} - 1)/(k_{56} - \omega^2 m_{56}), \quad \psi_{j6} = \lambda_j/(k_{64} - \omega^2 m_{64}) \end{aligned} \tag{38}$$

The coefficients d_j in Eq. (35) are obtained by the application of the boundary conditions that $\{V\}$ must satisfy. These boundary conditions, consistent with the boundary conditions on $\{W\}$, are determined according to Eq. (34). The boundary conditions corresponding to each of the standard beam end conditions, elaborated in Eqs. (26–28), are

Fixed end:

$$V_3 = V_5 = V_6 = 0$$

or

$$\sum_{j=1}^{6} d_j\psi_{ji}e^{-\lambda_j x} = 0, \qquad i = 3, 5, 6 \tag{39}$$

Hinged end:

$$V_2 = V_4 = V_5 = 0$$

or

$$\sum_{j=1}^{6} d_j\psi_{ji}e^{-\lambda_j x} = 0, \qquad i = 2, 4, 5 \tag{40}$$

Free end:

$$V_1 = 0, \quad V_4 + F_4V_5 = 0, \quad V_2 + F_2V_5 = 0$$

or

$$\begin{aligned} &\sum_{j=1}^{6} d_j\psi_{j1}e^{-\lambda_j x} = 0 \\ &\sum_{j=1}^{6} d_{ji}(\psi_{ji} - F_i\psi_{j5})e^{-\lambda_j x} = 0, \qquad i = 2, 4 \end{aligned} \tag{41}$$

In Eqs. (39–41), x is again equal to 0 or 1. A pair of equations (39–41) will provide a set of six homogeneous simultaneous equations for calculating the coefficients d_j as

$$[\Gamma]\{d\} = \{0\} \tag{42}$$

where d_j are the elements of $\{d\}$, and the elements of $[\Gamma]$ are defined by the coefficients of the two sets of equations (39–41), whichever are applicable to the beam boundary conditions. For a nontrivial solution, the determinant of $[\Gamma]$ must be zero. This provides the characteristic equation. The roots of this equation will be identical to the roots of the characteristic equation obtained from Eq. (30). Thus, knowing the frequency ω_n, one only needs to solve Eq. (42) for the coefficients d_{nj}. These coefficients along with eigenvector $\{\psi_{nj}\}$ define the adjoint eigenfunctions as

$$\{V_n\} = \sum_{j=1}^{6} d_{nj}\{\psi_{nj}\}e^{-\lambda_{nj}x} \tag{43}$$

It is quite simple to show that these eigenfunctions $\{W_n\}$ and $\{V_m\}$ satisfy the following biorthonormality conditions:

$$\int_0^1 \{V_m\}^T[M]\{W_n\}\,dx = -\delta_{mn} \tag{44}$$

$$\int_0^1 \{V_m\}^T(\{W_n'\} - [K]\{W_n\})\,dx = \omega_m^2\delta_{mn} \tag{45}$$

Equation (44) also serves as a normalization scheme for one or both of the eigenfunctions. We will now use these eigenfunctions to obtain a modal solution of the forced vibration problem of Eq. (16).

Response Analysis

We express the response vector $\{w\}$ in terms of the eigenfunction $\{W_n\}$ as in Eq. (32). Substituting this in Eq. (16) and premultiplying by the adjoint eigenfunction $\{V_m\}^T$ and integrating over the domain, we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} \int_0^1 q_n(t)\{V_m\}^T(\{W_n'\} - [K]\{W_n\})\,dx \\ &\quad - \int_0^1 \ddot{q}_n(t)\{V_m\}^T[M]\{W_n\}\,dx = \int_0^1 \{V_m\}^T\{F\}\,dx \end{aligned} \tag{46}$$

Using the orthogonality conditions in Eqs. (44) and (45), we obtain a decoupled equation for the mth principal coordinate as

$$\ddot{q}_m(t) + \omega_m^2 q_m(t) = \int_0^1 V_{m5}f(x,t)\,dx \tag{47}$$

where V_{m5} is the fifth element of $\{V_m\}$.

The solution of Eq. (47) substituted in Eq. (32) provides the time variation of the response vector $\{w\}$ as

$$\{w(x)\} = \sum_{m=1}^{\infty} \{W_m(x)\}\int_0^t h_m(t-\tau)\int_0^1 V_{m5}f(\nu,\tau)\,d\nu\,d\tau \tag{48}$$

where $h_m(t-\tau)$ is the impulse response function of Eq. (47). Also, a response quantity $R(x,t)$ linearly related to $w(x)$, such as a stress, can be similarly defined as

$$R(x,t) = \sum_{n=1}^{\infty} \rho_n(x)\int_0^t h_n(t-\tau)\int_0^1 V_{n5}(\nu)f(\nu,\tau)\,d\nu\,d\tau \tag{49}$$

where $\rho_n(x)$ is the nth modal response of the response quantity $R(x,t)$. For example, for the normal stress in a layer, this quantity is defined as

$$\rho_n = E_1 \frac{h}{L} Q_{11}^{*} z\left[\left(1-\frac{4}{3}z^2\right)W_{n6} - W_{n3}\right] \tag{50}$$

where W_{nj} is the jth element of $\{W_n\}$, and Q_{11}^{*} pertains to the layer where the stress is being calculated.

Random Response

For a random lateral load $f(x,t)$, the response $R(x,t)$ will be a random process. We will consider $f(x,t)$ to be a zero mean stationary random process with correlation function $R_{ff}(x_1,t_1;\,x_2,t_2)$. For this, the mean of $R(x,t)$ will also be zero and its correlation function $R_{RR}(x_1,t_1;\,x_2,t_2)$ can be defined as[20]

$$R_{RR}(x_1,t_1;x_2,t_2) = \sum_m^{\infty}\sum_n^{\infty} \rho_m(x_1)\rho_n(x_2)\int_0^{t_1}\!\!\int_0^{t_2} h_m(t_1-\tau_1)h_n(t_2-\tau_2)$$
$$\times \int_0^1\!\!\int_0^1 V_{m5}(\nu_1)V_{n5}(\nu_2)R_{ff}(\nu_1,\tau_1;\,\nu_2,\tau_2)\,\mathrm{d}\nu_1\,\mathrm{d}\nu_2\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \tag{51}$$

Equation (51) can be used to obtain the stationary mean square value of $R(x,t)$ in terms of spectral density function of $f(x,t)$ as

$$E[R^2(x)] = \sum_m^{\infty}\sum_n^{\infty}\rho_m(x)\rho_n(x)\int_{\infty}^{\infty}H_m(\omega)H_n^{*}(\omega)$$
$$\times \int_0^1\!\!\int_0^1 \Phi_{ff}(\omega,\nu_1,\nu_2)V_{m5}(\nu_1)V_{n5}(\nu_2)\,\mathrm{d}\nu_1\,\mathrm{d}\nu_2\,\mathrm{d}\omega \tag{52}$$

where $H_m(\omega)$ is the frequency response function of Eq. (47), an asterisk denotes the complex conjugate, and $\Phi_{ff}(\omega,\nu_1,\nu_2)$ is defined as

$$\Phi_{ff}(\omega,\nu_1,\nu_2) = \frac{1}{2\pi}\int_{\infty}^{\infty} R_{ff}(\nu_1,t_1;\,\nu_2,t_2)e^{-i\omega(t_2-t_1)}\,\mathrm{d}(t_2-t_1) \tag{53}$$

Equation (52) has been used to obtain the numerical mean square values of various response quantities in the beam for a stochastic model of load $f(x,t)$.

Numerical Results

In this section, we present some numerical results obtained for a symmetric cross-ply (0, 90, 90, 0) AS/3501 graphite-epoxy laminated beam with the following material properties: $E_1 = 2.1\times10^7$ N/m^2, $E_2 = 1.4\times10^6$ N/m^2, $G_{12} = G_{13} = 6\times10^5$ N/m^2, $G_{23} = 5\times10^5$ N/m^2, and $\nu_{12} = 0.3$. To assess the importance of shear deformations, the numerical results for various response quantities obtained by the classical beam theory and the first- and third-order shear deformation beam theories have been compared. In the first-order theory, a shear correction factor of 5/6 is used. Also, beams with clamped-clamped (CC), clamped-supported (CS), clamped-free (CF or cantilever), and simply supported (SS) boundary conditions have been considered to calculate the response by the proposed modal approach. For the forced vibration analysis, the standard deviations of the deflection and normal stress at critical locations of the beams have been obtained for random loading characterized by the following correlation function:

$$R_{ff}(\hat{x}_1,\hat{t}_1;\,\hat{x}_2,\hat{t}_2) = \sigma_0\exp(-\alpha|\hat{t}_2-\hat{t}_1| - \beta|\hat{x}_2-\hat{x}_1|) \tag{54}$$

For numerical calculations, the temporal and spatial correlation parameters have been taken as $\alpha = 0.1/T$ and $\beta = 0.05/L$. The random response results have been plotted in their nondimensional forms.

The percent differences in the fundamental frequencies of the beams with different boundary conditions calculated by the classical and third-order theories are shown in Fig. 2a and those calculated by the first- and third-order theories are shown in Fig. 2b. The third-order theory results are taken as the baseline for calculating the percent difference. For beams

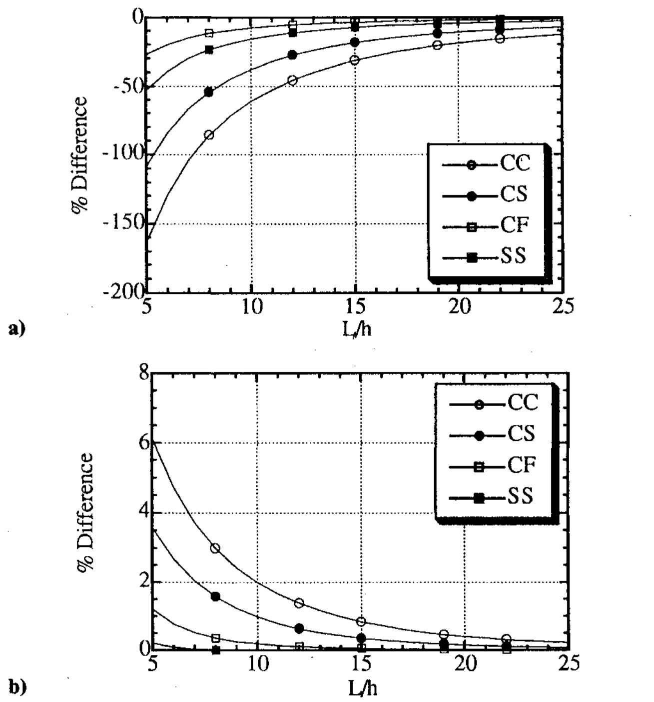

Fig. 2 Percent difference in the fundamental frequencies of beams with various boundary conditions: a) classical and third-order theories; b) first-order and third-order theories.

with small L/h ratios, very large differences between the results of the third and classical theories are noted in Fig. 2a, indicating the effect of the shear deformation on the accuracy of the calculated frequency. The CC beams are seen to be affected the most and the cantilever beams the least. That is, the more constrained a beam, the larger the difference. Because of permitting shear deformation in the third-order theory, the beams become more flexible and, thus, the frequencies calculated with this theory are smaller than those calculated with the classical approach. The difference between the results of the first-order and third-order theories, as shown in Fig. 2b, are, however, not as large as those in Fig. 2a. Also, the first-order theory renders the beam more flexible than the third-order theory. Again the CC beam shows the largest difference.

In Fig. 3, we show the standard deviations of the deflection of beams with various boundary conditions. For the CC, CS, and SS beams, the deflection standard deviations have been calculated at the midspan, whereas in the cantilever beam, it has been calculated at the free end. It is seen that the first- and third-order theories predict similar responses but they differ significantly from the response predicted by the classical beam theory, especially for beams with small L/h ratios. The percent differences between the standard deviation values calculated by the classical and third-order theories are shown in Fig. 4a, and those between the values calculated by the third- and first-order theories are shown in Fig. 4b. Again for the beams with small L/h ratios, large differences are seen between the classical and third-order theory results (Fig. 4a). Except for the CC and CS beams, the differences in the results of the first- and third-order theory results are not large, even for small L/h ratios. Also in Fig. 4a, the CC beam has the largest difference and the cantilever beam the smallest. Thus, the response of more restrained beams is again seen to be affected the most by shear deformations.

Figures 5a and 5b drawn at the fixed end for the CC and cantilever beams, respectively, show the variation of the standard deviation of the normal stress across the depth obtained for three L/h ratio beams. These results are for the third-order theory. Because of symmetry, the variation only over the upper half of the cross section is being shown. In CC beams with very low L/h ratios, an unusual distribution pattern is observed. The numerical results presented by Bhimaraddi[21] and Di Sciuva[22] for static analysis of plates obtained by using higher-order shear deformation theories and the elasticity solution do seem to corroborate the variation of standard deviation shown in Fig. 5a. In their numerical examples, the stress in the top lamina is seen to decrease from a positive maximum at the top to a negative minimum value at the bottom of the lamina. Since the standard deviation (always a positive quantity) is directly proportional to the magnitude of the stress at a level, it shows high values at the top and bottom of the lamina in our case. In the cantilever beam, on the other hand, this unusual pattern is not as pronounced as in the CC beam. Also, as the L/h ratio is increased, this effect disappears and the variation of the normal stress reverts back to its usual pattern both in the CC and cantilever beams. This unusual fixed-end effect also disappears fast as one moves away from the fixed end, as is shown in Figs. 6a and 6b which depict the standard deviations of the normal stresses at $\hat{x} = 0.05L$.

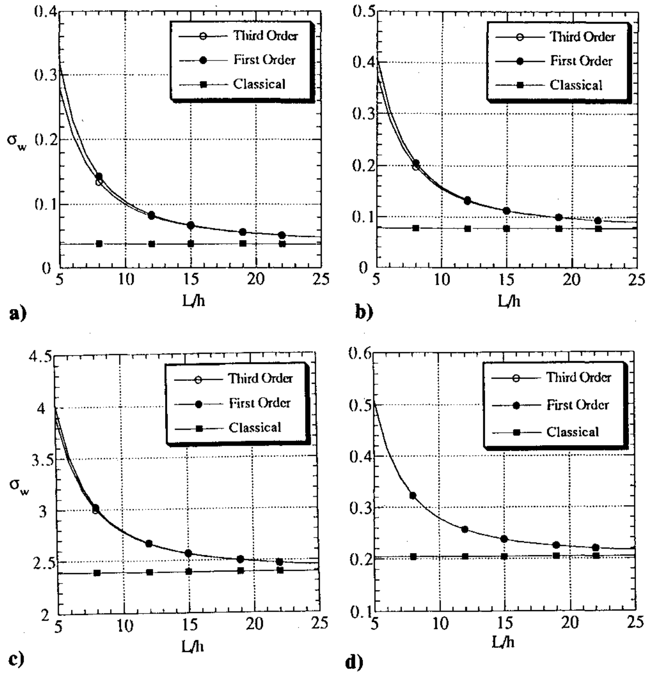

Fig. 3 Standard deviations of the normalized deflections calculated by different theories: a) clamped-clamped beam; b) clamped-supported beam; c) cantilever beam; d) simply supported beam.

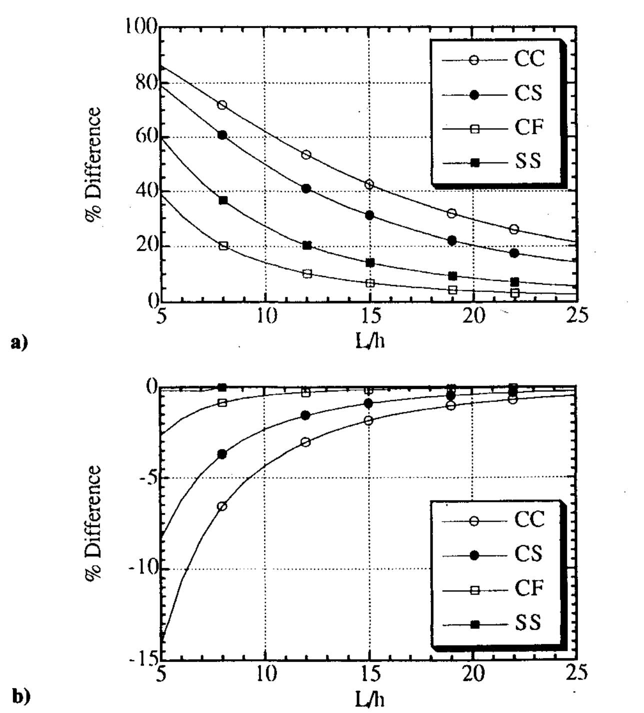

Fig. 4 Percent difference in the deflection standard deviations of beams with various boundary conditions: a) classical and third-order theories; b) first-order and third-order theories.

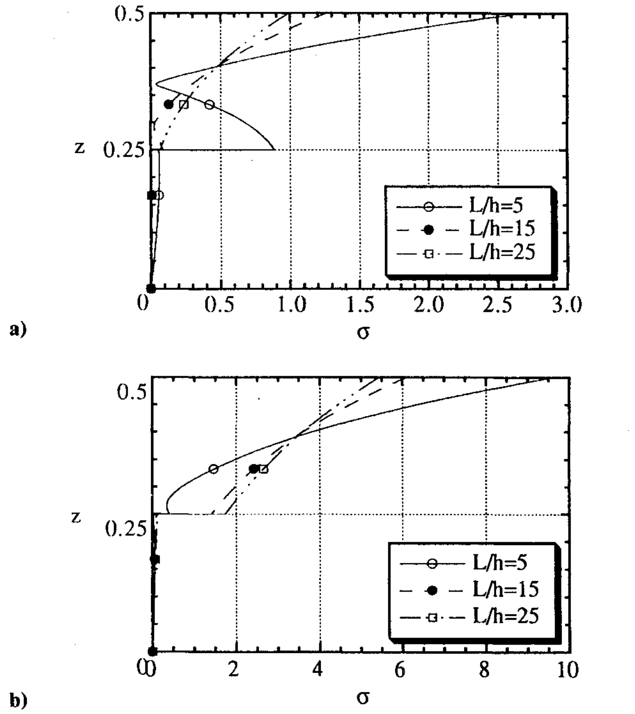

Fig. 5 Distribution of normal stress standard deviation across the depth at $x = 0$ for beams with three different L/h ratios (third-order theory): a) clamped-clamped beam; b) cantilever beam.

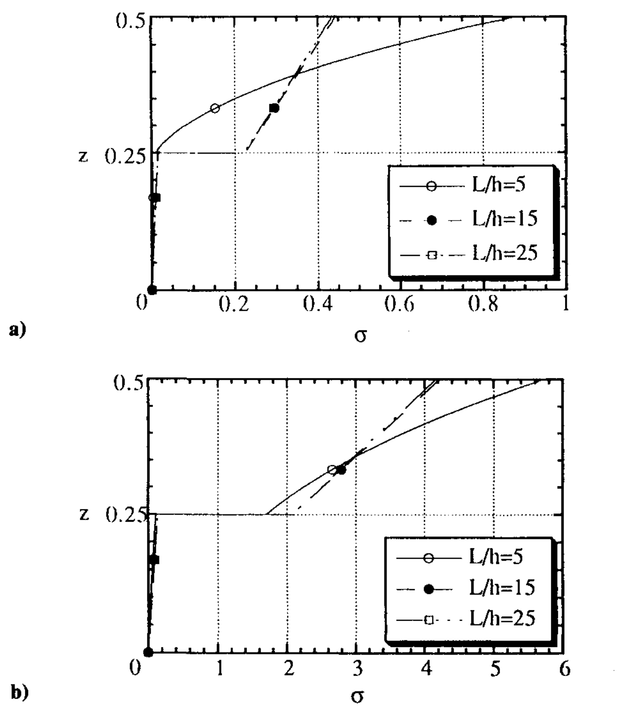

Fig. 6 Distribution of normal stress standard deviation across the depth at $x = 0.05$ for beams with three different L/h ratios (third-order theory): a) clamped-clamped beam; b) cantilever beam.

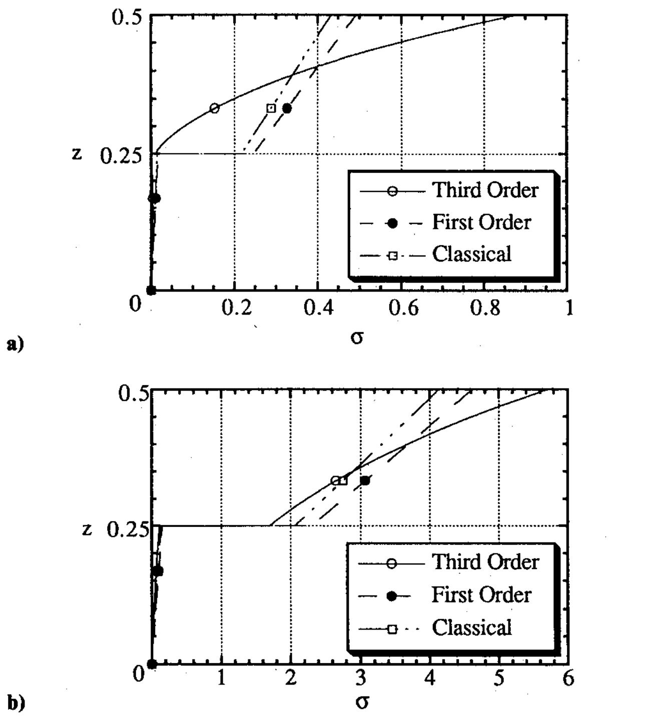

Fig. 7 Distribution of normal stress standard deviation across the depth calculated with various theories ($x = 0.05$, $L/h = 5$): a) clamped-clamped beam; b) cantilever beam.

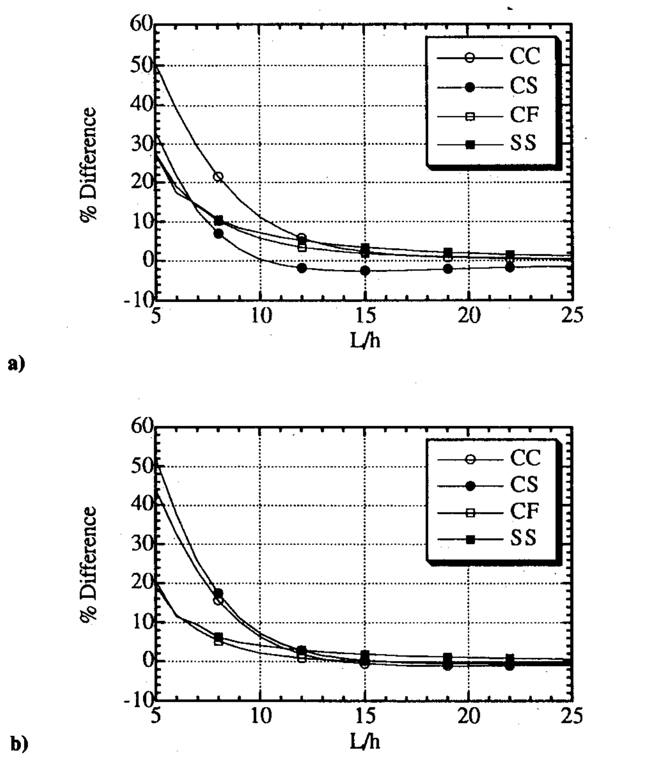

Fig. 8 Percent difference in the standard deviations of the normal stress of beams with various boundary conditions: a) classical and third-order theories; b) first-order and third-order theories.

In Figs. 7a and 7b, drawn for the CC and cantilever beams, respectively, we compare the standard deviations of the normal stress across the depth obtained at $\hat{x} = 0.05L$ with the three theories. The L/h ratio of the beams is 5. It is noted that, for the CC beam, the results for the classical and first-order theories are similar but they are different from those of the third-order theory. For the cantilever beam, this difference is not that large. To show this quantitatively for all four beams, we plot the percent difference in the standard deviation values calculated by the classical and third-order theories in Fig. 8a and those calculated by first-order and third-order theories in Fig. 8b. In the CC, CS, and cantilever beams, the maximum normal stress at $\hat{x} = 0.05L$ has been considered, whereas in the SS beam, the maximum stress at the midspan has been considered. It is noted that the difference between the stress standard deviations calculated by the third-order theory and the other two theories can be quite large for the beams with more boundary constraints and small L/h ratios. For large L/h ratios, of course, this difference becomes small as the shear deformation effects become relatively unimportant.

Concluding Remarks

For forced vibration analysis of laminated composite beams modeled by higher-order shear deformation theories, a generalized modal analysis approach is presented. The approach requires the calculation of the system frequencies and biorthonormal eigenfunctions of two adjoint boundary-value problems. The explicit expressions for the characteristic determinant (which is required to be solved to find system frequencies) for the adjoint boundary conditions and for calculating the biorthonormal eigenfunction are provided for symmetric cross-ply beams modeled by the third-order shear deformation theory. An approach to utilize these modal quantities to obtain the forced response of beams subjected to arbitrary loads is presented. The approach is used to calculate the random response of beams subjected to temporally and spatially correlated loads.

The numerical results for the system frequencies and the rms values of the deflection and stress responses are presented for beams with several different boundary conditions. To demonstrate the effect of the shear deformations on the frequencies and response, the numerical results obtained with the third- and first-order theories (which have been developed to include the effect of the shear deformations) are compared with the results obtained by the classical beam theory (which ignores the shear deformations). As expected, large differences are observed in the frequencies as well as the responses calculated by the classical and the third-order theory for short beams with small L/h ratios. It is also noted that whereas the first- and third-order theories seem to give similar results for the frequencies and deflection, there are significant differences in the distributions of the stress across the thickness obtained by the two theories. In general, the responses of beams with more end constraints (such as the CC and CS beams) are seen to be more affected by the shear deformation than the beams with less end constraints (such as SS and cantilever beams).

Appendix

Several elements of the $[K]$ and $[M]$ matrices in Eq. (16) are zero; the remaining nonzero k_{ij} and m_{ij} elements and the forcing function $f(x,t)$ are defined as follows:

$$f(x,t) = \frac{F(x,t)}{D_2}, \qquad F(x,t) = \frac{\hat{f}(\hat{x},\hat{t})L^3}{AE_1h^2} \tag{A1}$$

$$k_{12} = k_{23} = k_{35} = k_{46} = 1$$

$$k_{65} = \left(A_3 - \frac{4}{3}A_5\right)\Big/D_1, \quad k_{64} = (B_1 - 8B_3 + 16B_5)/D_1$$

$$k_{56} = k_{64}\left(A_3 - \frac{4}{3}A_5\right)\Big/D_2$$

$$m_{62} = -\left(\frac{h}{L}\right)^2\left(I_2 - \frac{4}{3}I_4\right)\Big/D_1$$

$$m_{64} = \left(\frac{h}{L}\right)^2\left(I_2 - \frac{8}{3}I_4 + \frac{16}{9}I_6\right)\Big/D_1$$

$$m_{51} = -1/D_2, \quad m_{53} = \left[\left(\frac{h}{L}\right)^2 I_2 + \left(A_3 - \frac{4}{3}A_5\right)m_{62}\right]\Big/D_2$$

$$m_{56} = \left[\left(\frac{h}{L}\right)^2\left(I_2 - \frac{4}{3}I_4\right) - \left(A_3 - \frac{4}{3}A_5\right)m_{64}\right]\Big/D_2$$

$$D_1 = A_3 - \frac{8}{3}A_5 + \frac{16}{9}A_7, \quad D_2 = A_3 - k_{65}\left(A_3 - \frac{4}{3}A_5\right) \qquad \text{(A2)}$$

where

$$A_k = \sum_{j=1}^{N} Q_{11}^*(Z_j^k - Z_{j-1}^k)/k; \qquad k = 1, \ldots, 7$$

$$B_k = \sum_{j=1}^{N} Q_{55}^*(Z_j^k - Z_{j-1}^k)/k; \qquad k = 1, 3, 5$$

$$I_k = \{(1/2)^{k+1} - (-1/2)^{k+1}\}/(k+1); \qquad k = 0, \ldots, 6$$

$$Q_{11}^* = \{Q_{11}\cos^4\theta_k + Q_{22}\sin^4\theta_k + 2(Q_{12} + 2Q_{66})\sin^2\theta_k\cos^2\theta_k\}/E_1$$

$$Q_{55}^* = (G_{13}\cos^2\theta_k + G_{23}\sin^2\theta_k)/E_1$$

$$Q_{11} = E_1/(1 - \nu_{12}\nu_{21}), \qquad Q_{22} = E_2/(1 - \nu_{12}\nu_{21})$$

$$Q_{12} = \nu_{12}E_2/(1 - \nu_{12}\nu_{21}), \qquad Q_{66} = G_{12} \qquad \text{(A3)}$$

where N is the number of layers in the laminate; θ the angle that the strong direction of the kth laminate makes with the x axis; E_1 and E_2 the Young's moduli in the strong and weak directions of a lamina, respectively; G_{12}, G_{13}, and G_{23} the shear moduli; and ν_{12} and ν_{21} the Poisson's ratios.

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